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LINEAR ALGEBRA

Inderjit singh

Founder of Siri Krishana SKM Academy

LINEAR ALGEBRA

A Complete Handbook With Counter Example For

CSIR-NET,IIT-JAM,GATE, TIFR,NBHM,MSc. Ent...

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Inderjit Singh

Founder of Siri Krishana SKM Academy

Linear Algebra

A complete handbook with counter examples for

CSIR-NET, GATE, IIT-JAM, M.Sc. Ent.

Inderjit Singh

Siri Krishana Math (SKM) Academy

Inderjit Singh

Linear Algebra

First edition

Preface

This book is intended to serve as a basic course in Linear Algebra and aims at providing adequate preparation to all NET, GATE, IIT-JAM aspirants, for M.Sc. entrance and other related competitive exams. The content is divided into seven chapters. The style and structure of Linear Algebra are designed to help students learn the core concepts and associated techniques in Algebra. Providing a fuller and richer amount of material than time allows in a lecture, this text presents topicwise theory, then exercise and after that solved questions from previous years papers, true/false, numerical answer type (NAT) and single/multiple correct questions for more and more practice.

In the first chapter, we included approximate 300 matrices and their types and properties, some special matrices and about limit of a matrix.

The second chapter includes rank and system of linear equations (both homogeneous and non homogeneous).

The third chapter begins with definitions and examples of vector spaces, then basis and dimension of vector spaces, row space, column space, complementary subspace and co-ordinate vectors.

The fourth chapter begins with definitions and examples of linear transformation, their construction, range space and null space, matrix representation of linear transformation.

The fifth chapter is devoted to study the diagonalization, characteristic polynomial, minimal polynomial of matrices and invariant and cyclic subspaces.

The sixth chapter covers Jordan canonical form, bilinear form, quadratic form, Hermitian form and their definiteness.

In seventh chapter, students can study about inner product space, normed vector space, orthogonality and projection of vectors.

Finally, both readable and mathematically interesting, the text also helps students learn the art of constructing mathematical arguments. Overall, the students discover how mathematics proceeds and how to use techniques that mathematicians actually employ.

Also, I special thanks to Ms. Komal (Assistant Professor in Dayanand Mahila Mahavidyalaya, Kurukshetra), Ms. Himani Sharma (Research Scholar in Thapar University, Patiala), Ms. Tanvi Singla (Research Scholar in Thapar University, Patiala) and Ms. Neha Rani (HPSC Assistant Professor of Mathematics) in editing of this book.

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CSIR-UGC National Eligibility Test (NET) for Junior Research Fellowship and Lecturer-ship

COMMON SYLLABUS FOR PART 'B' AND 'C'

MATHEMATICAL SCIENCES

UNIT – 1

Analysis: Elementary set theory, finite, countable and uncountable sets, Real number system as a complete ordered field, Archimedean property, supremum, infimum.

Sequences and series, convergence, limsup, liminf. Bolzano Weierstrass theorem, Heine Borel theorem.

Continuity, uniform continuity, differentiability, mean value theorem.

Sequences and series of functions, uniform convergence.Riemann sums and Riemann integral, Improper Integrals.

Monotonic functions, types of discontinuity, functions of bounded variation, Lebesgue measure, Lebesgue integral.

Functions of several variables, directional derivative, partial derivative, derivative as a linear transformation, inverse and implicit function theorems.

Metric spaces, compactness, connectedness. Normed linear Spaces. Spaces of continuous functions as examples.

Linear Algebra: Vector spaces, subspaces, linear dependence, basis, dimension, algebra of linear transformations.

Algebra of matrices, rank and determinant of matrices, linear equations.

Eigenvalues and eigenvectors, Cayley-Hamilton theorem.

Matrix representation of linear transformations. Change of basis, canonical forms, diagonal forms, triangular forms, Jordan forms.

Inner product spaces, orthonormal basis.

Quadratic forms, reduction and classification of quadratic forms

UNIT – 2

Complex Analysis: Algebra of complex numbers, the complex plane, polynomials, power series, transcendental functions such as exponential, trigonometric and hyperbolic functions.
Analytic functions, Cauchy-Riemann equations.

Contour integral, Cauchy's theorem, Cauchy's integral formula, Liouville's theorem, Maximum modulus principle, Schwarz lemma, Open mapping theorem.

Taylor series, Laurent series, calculus of residues.

Conformal mappings, Mobius transformations.

Algebra: Permutations, combinations, pigeon-hole principle, inclusion-exclusion principle, derangements.

Fundamental theorem of arithmetic, divisibility in Z, congruences, Chinese Remainder Theorem, Euler's Ø-function, primitive roots.

Groups, subgroups, normal subgroups, quotient groups, homomorphisms, cyclic groups, permutation groups, Cayley's theorem, class equations, Sylow theorems.

Rings, ideals, prime and maximal ideals, quotient rings, unique factorization domain, principal ideal domain, Euclidean domain.

Polynomial rings and irreducibility criteria.

Fields, finite fields, field extensions, Galois Theory.

Topology: basis, dense sets, subspace and product topology, separation axioms, connectedness and compactness.

UNIT – 3

Ordinary Differential Equations (ODEs):
Existence and uniqueness of solutions of initial value problems for first order ordinary differential equations, singular solutions of first order ODEs, system of first order ODEs.

General theory of homogenous and non-homogeneous linear ODEs, variation of parameters, Sturm-Liouville boundary value problem, Green's function.

Partial Differential Equations (PDEs):
Lagrange and Charpit methods for solving first order PDEs, Cauchy problem for first order PDEs.

Classification of second order PDEs, General solution of higher order PDEs with constant coefficients, Method of separation of variables for Laplace, Heat and Wave equations.

Numerical Analysis :

Numerical solutions of algebraic equations, Method of iteration and Newton-Raphson method, Rate of convergence, Solution of systems of linear algebraic equations using Gauss elimination and Gauss-Seidel methods, Finite differences, Lagrange, Hermite and spline interpolation, Numerical differentiation and integration, Numerical solutions of ODEs using Picard, Euler, modified Euler and Runge-Kutta methods .

Calculus of Variations:

Variation of a functional, Euler-Lagrange equation, Necessary and sufficient conditions for extrema. Variational methods for boundary value problems in ordinary and partial differential equations.

Linear Integral Equations:

Linear integral equation of the first and second kind of Fredholm and Volterra type, Solutions with separable kernels. Characteristic numbers and eigenfunctions, resolvent kernel.

Classical Mechanics:

Generalized coordinates, Lagrange's equations, Hamilton's canonical equations, Hamilton's principle and principle of least action, Two-dimensional motion of rigid bodies, Euler's dynamical equations for the motion of a rigid body about an axis, theory of small oscillations.

UNIT – 4

Descriptive statistics, exploratory data analysis

Sample space, discrete probability, independent events, Bayes theorem. Random variables and distribution functions (univariate and multivariate); expectation and moments. Independent random variables, marginal and conditional distributions. Characteristic functions. Probability inequalities (Tchebyshef, Markov, Jensen). Modes of convergence, weak and strong laws of large numbers, Central Limit theorems (i.i.d. case).

Markov chains with finite and countable state space, classification of states, limiting behaviour of n-step transition probabilities, stationary distribution, Poisson and birth-and-death processes.

Standard discrete and continuous univariate distributions. sampling distributions, standard errors and asymptotic distributions, distribution of order statistics and range.

Methods of estimation, properties of estimators, confidence intervals. Tests of hypotheses: most powerful and uniformly most powerful tests, likelihood ratio tests. Analysis of discrete data and chi-square test of goodness of fit. Large sample tests.

Simple nonparametric tests for one and two sample problems, rank correlation and test for independence. Elementary Bayesian inference.

Gauss-Markov models, estimability of parameters, best linear unbiased estimators, confidence intervals, tests for linear hypotheses. Analysis of variance and covariance. Fixed, random and mixed effects models. Simple and multiple linear regression. Elementary regression diagnostics. Logistic regression.

Multivariate normal distribution, Wishart distribution and their properties. Distribution of quadratic forms. Inference for parameters, partial and multiple correlation coefficients and related tests. Data reduction techniques: Principle component analysis, Discriminant analysis, Cluster analysis, Canonical correlation.

IIT-JAM MATHEMATICS (MA) SYLLABUS

Sequences and Series of Real Numbers: Sequence of real numbers, convergence of sequences, bounded and monotone sequences, convergence criteria for sequences of real numbers, Cauchy sequences, subsequences, Bolzano-Weierstrass theorem. Series of real numbers, absolute convergence, tests of convergence for series of positive terms – comparison test, ratio test, root test; Leibniz test for convergence of alternating series.

Functions of One Real Variable: Limit, continuity, intermediate value property, differentiation, Rolle's Theorem, mean value theorem, L'Hospital rule, Taylor's theorem, maxima and minima.

Functions of Two or Three Real Variables: Limit, continuity, partial derivatives, differentiability, maxima and minima.

Integral Calculus: Integration as the inverse process of differentiation, definite integrals and their properties, fundamental theorem of calculus. Double and triple integrals, change of order of integration, calculating surface areas and volumes using double integrals, calculating volumes using triple integrals.

Differential Equations: Ordinary differential equations of the first order of the form y'=f(x,y), Bernoulli's equation, exact differential equations, integrating factor, orthogonal trajectories, homogeneous differential equations, variable separable equations, linear differential equations of second order with constant coeffcients, method of variation of parameters, Cauchy-Euler equation.

Vector Calculus: Scalar and vector fields, gradient, divergence, curl, line integrals, surface integrals, Green, Stokes and Gauss theorems.

Group Theory: Groups, subgroups, Abelian groups, non-Abelian groups, cyclic groups, permutation groups, normal subgroups, Lagrange's Theorem for finite groups, group homomorphisms and basic concepts of quotient groups.

Linear Algebra: Finite dimensional vector spaces, linear independence of vectors, basis, dimension, linear transformations, matrix representation, range space, null space, rank-nullity theorem. Rank and inverse of a matrix, determinant, solutions of systems of linear equations, consistency conditions, eigenvalues and eigenvectors for matrices, Cayley-Hamilton theorem.

Real Analysis: Interior points, limit points, open sets, closed sets, bounded sets, connected sets, compact sets, completeness of R. Power series (of real variable), Taylor's series, radius and interval of convergence, term-wise differentiation and integration of power series.

IIT-GATE MATHEMATICS (MA) SYLLABUS

Calculus: Functions of two or more variables, continuity, directional derivatives, partial derivatives, total derivative, maxima and minima, saddle point, method of Lagrange's multipliers; Double and Triple integrals and their applications to area, volume and surface area; Vector Calculus: gradient, divergence and curl, Line integrals and Surface integrals, Green's theorem, Stokes' theorem, and Gauss divergence theorem.

Linear Algebra: Finite dimensional vector spaces over real or complex fields; Linear transformations and their matrix representations, rank and nullity; systems of linear equations, characteristic polynomial, eigenvalues and eigenvectors, diagonalization, minimal polynomial, Cayley-Hamilton Theorem, Finite dimensional inner product spaces, Gram-Schmidt orthonormalization process, symmetric, skew-symmetric, Hermitian, skew-Hermitian, normal, orthogonal and unitary matrices; diagonalization by a unitary matrix, Jordan canonical form; bilinear and quadratic forms.

Real Analysis: Metric spaces, connectedness, compactness, completeness; Sequences and series of functions, uniform convergence, Ascoli-Arzela theorem;Weierstrass approximation theorem; contraction mapping principle, Power series; Differentiation of functions of several variables, Inverse and Implicit function theorems; Lebesgue measure on the real line, measurable functions; Lebesgue integral, Fatou's lemma, monotone convergence theorem, dominated convergence theorem.

Complex Analysis: Functions of a complex variable: continuity, differentiability, analytic functions, harmonic functions; Complex integration: Cauchy's integral theorem and formula; Liouville's theorem, maximum modulus principle, Morera's theorem; zeros and singularities; Power series, radius of convergence, Taylor's series and Laurent's series; Residue theorem and applications for evaluating real integrals; Rouche's theorem, Argument principle, Schwarz lemma; Conformal mappings, Mobius transformations.

Ordinary Differential equations: First order ordinary differential equations, existence and uniqueness theorems for initial value problems, linear ordinary differential equations of higher order with constant coefficients; Second order linear ordinary differential equations with variable coefficients; Cauchy-Euler equation, method of Laplace transforms for solving ordinary differential equations, series solutions (power series, Frobenius method); Legendre and Bessel functions and their orthogonal properties; Systems of linear first order ordinary differential equations, Sturm's oscillation and separation theorems, Sturm-Liouville eigenvalue problems, Planar autonomous systems of ordinary differential equations: Stability of stationary points for linear systems with constant coefficients, Linearized stability, Lyapunov functions.

Algebra: Groups, subgroups, normal subgroups, quotient groups, homomorphisms, automorphisms; cyclic groups, permutation groups,Group action**,**Sylow's theorems and their applications; Rings, ideals, prime and maximal ideals, quotient rings, unique factorization domains, Principle ideal domains, Euclidean domains, polynomial rings, Eisenstein's irreducibility criterion; Fields, finite fields, field extensions,algebraic extensions, algebraically closed fields.

Functional Analysis: Normed linear spaces, Banach spaces, Hahn-Banach theorem, open mapping and closed graph theorems, principle of uniform boundedness; Inner-product spaces, Hilbert spaces, orthonormal bases, projection theorem,Riesz representation theorem, spectral theorem for compact self-adjoint operators.

Numerical Analysis: Systems of linear equations: Direct methods (Gaussian elimination, LU decomposition, Cholesky factorization), Iterative methods (Gauss-Seidel and Jacobi) and their convergence for diagonally dominant coefficient matrices; Numerical solutions of nonlinear equations: bisection method, secant method, Newton-Raphson method, fixed point iteration; Interpolation: Lagrange and Newton forms of interpolating polynomial, Error in polynomial interpolation of a function; Numerical differentiation and error, Numerical integration: Trapezoidal and Simpson rules, Newton-Cotes integration formulas, composite rules, mathematical errors involved in numerical integration formulae; Numerical solution of initial value problems for ordinary differential equations: Methods of Euler, Runge-Kutta method of order 2.

Partial Differential Equations: Method of characteristics for first order linear and quasilinear partial differential equations; Second order partial differential equations in two independent variables: classification and canonical forms, method of separation of variables for Laplace equation in Cartesian and polar coordinates, heat and wave equations in one space variable; Wave equation: Cauchy problem and d'Alembert formula, domains of dependence and influence, non-homogeneous wave equation; Heat equation: Cauchy problem; Laplace and Fourier transform methods.

Topology: Basic concepts of topology, bases, subbases, subspace topology, order topology, product topology, quotient topology, metric topology, connectedness, compactness, countability and separation axioms, Urysohn's Lemma.

Linear Programming: Linear programming models, convex sets, extreme points;Basic feasible solution,graphical method, simplex method, two phase methods, revised simplex method ; Infeasible and unbounded linear programming models, alternate optima; Duality theory, weak duality and strong duality; Balanced and unbalanced transportation problems, Initial basic feasible solution of balanced transportation problems (least cost method, north-west corner rule, Vogel's approximation method); Optimal solution, modified distribution method; Solving assignment problems, Hungarian method.

NBHM PHD. 2021 SYLLABUS

Since this is a Ph.D. Course, the syllabus is not well structured and predefined, instead, it covers topics from UG and PG levels of Mathematics. The question paper is divided into three sections. To know more about the outline of the syllabus and its sections check below:

Section A: Algebra Syllabus

NBHM Algebra syllabus covers multiple subtopics as mentioned below:
Abstract Algebra, Set Theory Binary, Operations Matrix Theory, Groups Algebra, Rings and Fields, Polynomials (Calculus, Matrix Polynomials, Roots)

Section B: Analysis Syllabus

Mathematics - Analysis Syllabus covers majorly three sections such as Complex; Functional; Real. Candidates are suggested to cover these topics thoroughly, as they are vast. Below mentioned is the NBHM Analysis Syllabus in detail:

Complex Analysis Syllabus
Polar Coordinates, Poles and Residues

Real Analysis Syllabus
Sequence and Limits, Metric Spaces, Series (Infinite Series Expansion, Taylor Series Expansion)

Functional Analysis Syllabus
Continuous Function, Maxima and Minima, Differential Function ,Defining Function

Section C: Geometry Syllabus

NBHM Geometry Syllabus contains mainly two sub-topics. Check below to know more:

Plane Algebraic Syllabus
Line, Circle, Ellipses, Elliptical Curve, Cubic Curves, Spheres (3D Shapes)

Algebraic Geometry Syllabus
Polar Coordinates, Cartesian Coordinates

TIFR

GS2021: SELECTION PROCESS FOR MATHEMATICS AND SYLLABUS

Selection process for admission in 2021 to the various programs in Mathematics at the TIFR cen- ters - namely, the PhD and Integrated PhD programs at TIFR, Mumbai as well as the programs conducted by TIFR CAM, Bengaluru and ICTS, Bengaluru - will be held in two stages.

Stage I. A nation-wide test will be conducted in various centers on March 7, 2021. For the PhD and Integrated PhD programs at the Mumbai Center, this test will comprise the entirety of Stage I of the evaluation process. For more precise details about Stage I of the selection process at other centers (TIFR CAM, Bengaluru, and ICTS, Bengaluru) we refer you to the websites of those centers.

The nation-wide test on March 7 will be an objective test of three hours duration, with 20 multiple choice questions and 20 true/false questions. The score in this test will serve as qualification marks for a student to progress to the second step of the evaluation process. The cut-off marks for a particular program will be decided by the TIFR center handling that program.

Additionally, some or all of the centers may consider the score in Stage I (in addition to the score in Stage II) towards making the final selection for the graduate program in 2021.

Stage II. The second stage of the selection process varies according to the program and the center. More details about this stage will be provided at a later date.

Syllabus for Stage I

Stage I of the selection process is mainly based on mathematics covered in a reasonable B.Sc. course. This includes:

Algebra: Definitions and examples of groups (finite and infinite, commutative and non-commutative), cyclic groups, subgroups, homomorphisms, quotients. Group actions and Sylow theorems. Defini- tions and examples of rings and fields. Integers, polynomial rings and their basic properties. Basic facts about vector spaces, matrices, determinants, ranks of linear transformations, characteristic and minimal polynomials, symmetric matrices. Inner products, positive definiteness.

Analysis: Basic facts about real and complex numbers, convergence of sequences and series of real and complex numbers, continuity, differentiability and Riemann integration of real valued functions defined on an interval (finite or infinite), elementary functions (polynomial functions, rational functions, exponential and log, trigonometric functions), sequences and series of functions and their different types of convergence.

Geometry/Topology: Elementary geometric properties of common shapes and figures in 2 and 3 dimensional Euclidean spaces (e.g. triangles, circles, discs, spheres, etc.). Plane analytic geometry

(= coordinate geometry) and trigonometry. Definition and basic properties of metric spaces, exam- ples of subset Euclidean spaces (of any dimension), connectedness, compactness. Convergence in metric spaces, continuity of functions between metric spaces.

General: Pigeon-hole principle (box principle), induction, elementary properties of divisibility, elementary combinatorics (permutations and combinations, binomial coefficients), elementary rea- soning with graphs, elementary probability theory.

Sample Questions for Stage I

The following are some sample questions for the online test that will be held on March 7. You can find some of the previous years' question papers at:
http://univ.tifr.res.in/gs2021/Prev QP/Prev QP.htm_

Sample multiple choice questions

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous bounded function. Then,
   a) f has to be uniformly continuous.
   b) there exists an $x \in \mathbb{R}$ such that $f(x) = x$
   c) f can not be increasing

d) $\lim_{x\to\infty} f(x)$ exists.

2. Define a function

$$f(x) = \begin{cases} x + x^2 \cos\left(\frac{\pi}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Consider the statements:

I. f is differentiable at $x = 0$ and $f'(0) = 1$.

II. f is differentiable everywhere and $f'(x)$ is continuous at $x = 0$.

III. f is increasing in a neighbourhood around $x = 0$.

IV. f is not increasing in any neighbourhood of $x = 0$.

Which one of the following combinations of the above statements is true ?

a) **I.** and **II**.

b) **I.** and **III.**

c) **II.** and **IV.**

d) **I.** and **IV.**

Sample true/false questions

1. **If A and B are 3×3 matrices and A is invertible, then there exists an integer n such that A+nB is invertible.**

2. **Let P be a degree 3 polynomial with complex coefficients such that the constant term is 2010. Then P has a root α with $|\alpha| > 10$.**

3. **The symmetric group S_5 consisting of permutations on 5 symbols has an element of order 6.**

4. **Suppose $f_n(x)$ is a sequence of continuous functions on the closed interval [0,1] converging to 0 pointwise. Then the integral**

$$\int_0^1 f_n(x)dx$$

converges to 0.

5. **There are n homomorphisms from the group $\mathbb{Z}/n\mathbb{Z}$ to the additive group of rationals $\mathbb{Q}$.**

6. **A bounded continuous function on $\mathbb{R}$ is uniformly continuous.**

CONTENTS

- Answer Key of IIT-JAM Linear Algebra from year 2005 to 2020_P617
- Complete Question Paper with Answer key of IIT-JAM 2021_P618 to P623
- Complete Question Paper with Answer key of GATE 2021_P624 to P631
- Complete Question Paper of TIFR 2021_P632 to P634
- Complete Question Paper of NBHM 2020_P635 to P638

Our next edition of this book will contain these topics also:

- Check matrix
- Unimodular integer matrix
- Cosine of a matrix
- Sine of a matrix
- Exponential function of a matrix
- Convergent matrix
- Limit of matrices
- Defective matrix
- Derivative matrix
- Integral matrix
- Dominating non negative matrix
- Dominating principal diagonal
- Northwest matrix
- Frobenius matrix
- Arithmetic matrix
- Geometric matrix
- Band matrix or banded matrix
- Bordered matrix
- Brownian matrix
- Controllability matrix
- Gram matrix
- Hamiltonian matrix
- Helmert matrix
- Hermite canonical form
- Loewner matrix
- Hessenberg matrix
- Hurwitz matrix
- Kronecker matrix
- Leontief matrix
- Leslie matrix
- Metzler matrix
- Minkowski-Metzler matrix
- Partitioned matrix
- Routh matrix
- Rational matrix
- Sylvester matrix
- Sparse matrix

- Similar matrices
- Harmonic matrix
- Bordered Gramian matrix
- Modal matrix
- Bidiagonal matrix
- Important definitions related to row vector and column vector
- Important definitions related to polynomial matrices
- Commutator of matrices and its results
- Modulus of a matrix and its results
- Important equivalent conditions based on Hermitian and Orthogonal matrices
- Triangularization
    - Triangularizable linear operator
    - Triangularizable matrix
    - Block diagonalization
    - Generalized eigenvector
    - Null space
    - Range space
    - Important results based on null space and range space
    - Important results based on Triangularization
    - Block upper triangular matrices
    - Eigenpair of a linear transformation
- Rational canonical form
    - Irreducible and reducible polynomial
    - Rational canonical form
    - T-invariant subspace
    - T-cyclic subspace
    - Important formulas for rational canonical form

1. MATRICES

1.1 Matrix

Let R be a commutative ring with unity and let m, n be non negative integers. An array of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n} = [a_{ij}]_{\substack{i=1,\dots,m \\ j=1,\dots,n}} = [a_{ij}]_{m\times n},$$

with $a_{ij} \in R, i = 1,2, \dots, m;\ j = 1,2, \dots, n$, is called a **matrix** of size $m \times n$ over R. The a_{ij}'s are called the **entries** or **coefficients** or $(i, j)^{th}$ element of the matrix.
The set of all such matrices is denoted by $M_{m\times n}(R)$ or $M^{m\times n}(R)$.
Strictly speaking, the rectangular array displayed above is not a matrix but it is a representation of a matrix.

Or

An m×n matrix over the field F is a function A from the set of pair of integers (i, j) ; $i = 1,2, \dots, m$, $j = 1,2, \dots, n$ into the field F.

Remarks 1.1.1

1. The m×n matrix is called **rectangle** if $m \neq n$ and **square** or **quadratic** if $m = n$.
2. In above definition, we assume that $1 \neq 0$ in R to exclude trivial case of the ring that contains only the zero element.
3. For $m = 0$ or $n = 0$, we obtain empty matrices of the size $m \times 0$, $0 \times n$ or 0×0, denoted by [ ].
4. In case of square matrices, the set of all such matrices is denoted by $M_{n\times n}(R)$ or $M_n(R)$ or $M^{n\times n}(R)$.

An $m \times n$ matrix $A = [a_{ij}]$ is

1. **Negative matrix** if $a_{ij} < 0$ for $i = 1, \dots, m, j = 1, \dots, n$.
2. **Positive matrix** if $a_{ij} > 0$ for $i = 1, \dots, m, j = 1, \dots, n$.
3. **Nonnegative matrix** if $a_{ij} \geq 0$ for $i = 1, \dots, m, j = 1, \dots, n$.
4. **Nonpositive matrix** if $a_{ij} \leq 0$ for $i = 1, \dots, m, j = 1, \dots, n$.
5. **Seminegative matrix** : if $a_{ij} \leq 0$ for $i = 1, \dots, m, j = 1, \dots, n$ and at least one element is strictly negative.
6. **Semipositive matrix** : if $a_{ij} \geq 0$ for $i = 1, \dots, m, j = 1, \dots, n$ and at least one element is strictly positive.
7. **Complex matrix** if all elements a_{ij} are complex numbers, i.e., $a_{ij} \in \mathbb{C}$.
8. **Real matrix** if all elements a_{ij} are real numbers, i.e., $a_{ij} \in \mathbb{R}$.
9. **Integer matrix** if elements a_{ij} are integers, i.e., $a_{ij} \in \mathbb{Z}$.

1.1.2 Dimension of a matrix : $m \times n$ is the dimension or **order** of a matrix with m rows and n columns. Sometimes, the degree of the characteristic polynomial of A is called its order. The number of columns of a matrix is its **column dimension** and the number of rows of a matrix is its **row dimension**.

1.1.3 Diagonal of a matrix : A diagonal of an $n \times n$ matrix $A = [a_{ij}]$ consists of all elements a_{ij} for which $i - j$ equals a given integer zero. Or

The elements $a_{ii}, i = 1, \ldots, n$, constitute the principal diagonal or main diagonal of an $n \times n$ matrix $A = [a_{ij}]$.

1.1.4 Secondary diagonal : The elements $a_{1+i,n-i}, i = 0, \ldots, n-1$, of an $n \times n$ matrix $A = [a_{ij}]$ constitute the secondary diagonal of A. In other words, the secondary diagonal is the diagonal from the upper right hand to the lower left hand corner of A.

1.1.5 Absolute value of a matrix : The absolute value of a matrix $A_{m\times n} = [a_{ij}]$ is defined as

$$|A|_{abs} \equiv \left[|a_{ij}|_{abs}\right] = \begin{bmatrix} |a_{11}|_{abs} & |a_{12}|_{abs} & \cdots & |a_{1n}|_{abs} \\ |a_{21}|_{abs} & |a_{22}|_{abs} & \cdots & |a_{2n}|_{abs} \\ \vdots & \vdots & & \vdots \\ |a_{m1}|_{abs} & |a_{m2}|_{abs} & \cdots & |a_{mn}|_{abs} \end{bmatrix}_{m\times n},$$

where $|c|_{abs}$ denotes the modulus of the complex number $c = c_1 + ic_2$ defined as $|c|_{abs} = \sqrt{c_1^2 + c_2^2} = \sqrt{c\bar{c}}$. Here, $\bar{c}$ is the complex conjugate of c.

1.1.6 Matrix function : A function $f : \mathbb{C}^{m\times n} \to \mathbb{C}$ (or $\mathbb{R}^{m\times n} \to \mathbb{R}$), $A \longmapsto f(A)$ is called a matrix function or function of a matrix. **e.g.,** the determinant and the trace, where $n = m$.

1.2 Submatrix

Let A be $m \times n$ matrix. Then a matrix obtained from A by deleting some rows or columns of A is called **submatrix** of A, i.e., a submatrix of a given matrix is an array lying in specified subsets of the rows and columns of the given matrix,

i.e., a matrix of order $p \times q$ given by $B = [a_{i_k j_r}]$ with $i_1 < \cdots < i_p$ and $j_1 < \cdots < j_q$ is a submatrix of the $m \times n$ matrix $A = [a_{ij}]$.

Example 1.2.1 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$. Submatrices of A are $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}, [1 \quad 2 \quad 3],$ $[4 \quad 5 \quad 6]$ etc.

1.2.2 Principal diagonal submatrix : The matrices of the form $\begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix}$ for $k = 1, \ldots, n-1$, are the principal diagonal submatrices of the $n \times n$ matrix $A = [a_{ij}]$.

Example 1.2.3 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times 3}$. Principal diagonal submatrices of A are $[1], [5], [9], \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix},$ $\begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$

1.2.4 Important formulas for submatrices

1. Let A be $m \times n$ matrix. Then, the number of submatrices of A is

i) if all the entries of A are distinct, then $(2^m - 1)(2^n - 1)$.
ii) if all the entries of A are same, then $m.n$.

2. Let A be $m \times n$ matrix. Then, the number of submatrices of order $p \times q$ is
   i) if all the entries of A are distinct, then ${}^mC_p.{}^nC_q$.
   ii) if all the entries of A are same, then each order submatrix is unique.
3. Let A be a square matrix of order n. Then the number of principal diagonal submatrices of A is
   i) if all the entries of A are distinct, then $2^n - 1$.
   ii) if all the entries of A are same, then n.
4. Let A be a square matrix of order n. Then the number of principal diagonal submatrices of order m is
   i) if all the entries of A are distinct, then ${}^nC_m$.
   ii) if all the entries of A are same, then unique.

1.3 Some Basic Matrix Definitions

1. **Row matrix** : A matrix having only one row is called a row matrix or a **row vector.** **e.g.,** $A = [1 \quad 2 \quad 3 \quad 6]$ is row matrix of order 1×4.
2. **Column matrix** : A matrix having only one column is called column matrix or a **column vector**. **e.g.,** $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a column vector of order 3×1.
3. **Diagonal matrix** : A square matrix $A = [a_{ij}]$ of order $n \times n$ is called diagonal matrix if $a_{ij} = 0$ for all $i \neq j$ and the elements a_{ij} for $i = j$ are called diagonal elements and the line along which they lie is called the principal diagonal. **e.g.,** $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a diagonal matrix and $a_{11} = 2$, $a_{22} = 3$, $a_{33} = 5$ are called diagonal elements.
   Or an $n \times n$ diagonal matrix whose diagonal elements are $d_1, d_2, \ldots, d_n$ can be represented as $\text{diag}(d_1, d_2, \ldots, d_n)$.
4. **Scalar matrix** : A square matrix $A = [a_{ij}]$ of order n is called scalar matrix if it is diagonal and all diagonal elements are equal, i.e., if for some $c \in \mathbb{C}$, $a_{ij} = \begin{cases} c & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$,
   i.e., $A = \text{diag}(c, \ldots, c) = cI_n$. **e.g.,** $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.
5. **Unit(Identity)matrix** : An $n \times n$ matrix $I_n = [a_{ij}]$ with $a_{ii} = 1$ for $i = 1, \ldots, n$ and $a_{ij} = 0$ for $i \neq j$ is an identity or unit matrix. **e.g.,** $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

6. **Reverse unit matrix** : The $n \times n$ matrix $A = [a_{ij}]$ with $a_{ij} = \begin{cases} 1 & \text{for } i + j = n + 1 \\ 0 & \text{otherwise} \end{cases}$ given by $\begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$ is called **reverse unit matrix** or an **exchange matrix** or **backward identity** or **flip matrix**.
7. **Null matrix** : An $m \times n$ matrix whose all elements are zero is called a null matrix or a **zero matrix**, denoted by $O_{m\times n}$ or simply by 0. **e. g.**, The matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is null matrix of order 2.
8. **Upper triangular matrix** : A square matrix is called upper triangular matrix if all the elements below the principal diagonal are zero, i.e., if $a_{ij} = 0$ for all $i > j$.
   e. g., The matrix $\begin{bmatrix} 1 & 3 & 8 \\ 0 & -1 & 7 \\ 0 & 0 & 0 \end{bmatrix}$ is upper triangular matrix and $\begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix}$ is unit upper triangular matrix.
   An upper triangular matrix of order n with $a_{ii} = 1$ for $i = 1,2,\dots,n$ and $a_{ij} = 0$ for $i > j$ is **unit upper triangular matrix.** And if $a_{ij} = 0$ for all $i \geq j$, then it is called **strictly upper triangular matrix.**
9. **Lower triangular matrix** : A square matrix is called lower triangular matrix if all the elements above the principal diagonal are zero, i.e., if $a_{ij} = 0$ for all $i < j$.
   e. g., The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 7 & 5 & 2 \end{bmatrix}$ is lower triangular matrix and $\begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 7 & 5 & 0 \end{bmatrix}$ is strictly lower triangular matrix.
   A lower triangular matrix of order n with $a_{ii} = 1$ for $i = 1,2,\dots,n$ and $a_{ij} = 0$ for $i < j$ is **unit lower triangular matrix**. And if $a_{ij} = 0$ for all $i \leq j$, then it is called **strictly lower triangular matrix.**
10. **Triangular matrix :** A square matrix in which all the elements below the principal diagonal or above the principal diagonal are zero is called triangular matrix. In other words, a matrix which is either upper triangular or lower triangular is called a triangular matrix. **e. g.,** $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & -4 & 1 \end{bmatrix}$.
    A unit lower or unit upper triangular matrix is a **unit triangular matrix**. And a matrix is **strictly triangular** if it is strictly lower triangular or strictly upper triangular.

1.4 Basic Operations on Matrices

1.4.1 Addition of matrices : The sum of two matrices is defined only if they have same order. Let A, B be two matrices of order $m \times n$. Then their sum $A + B$ is obtained by adding corresponding elements of A and B and has same order as of A and B.

Example 1.4.2 Let $A=\begin{bmatrix}1 & 2\\ 1 & -1\end{bmatrix}$ and $B=\begin{bmatrix}4 & 4\\ 2 & 0\end{bmatrix}$. Then $A+B=\begin{bmatrix}5 & 6\\ 3 & -1\end{bmatrix}$.

1.4.3 Properties of matrix addition :

Let A , B and C be matrices of order $m \times n$. Then

1. matrix addition is commutative, i.e., $A+B = B+A$.
2. matrix addition is associative, i.e., $(A+B)+C = A+(B+C)$.
3. the null matrix is additive identity for matrix addition, i.e., $A+O = A = O+A$, where O is null matrix of order $m \times n$.
4. for every matrix $A = [a_{ij}]_{m\times n}$, there exists a matrix $[-a_{ij}]_{m\times n}$ denoted by $-A$ such that $A + (-A) = O = (-A) + A$.
5. both cancellation laws hold in matrix addition,
   i.e., $A+B = A+C \Rightarrow B = C$ (left cancellation law)
   and $B+A = C+A \Rightarrow B = C$ (right cancellation law)
6. set of all $m \times n$ matrices forms an abelian group under addition.

1.4.4 Direct sum : The direct sum of two matrices $A_{m\times m} = [a_{ij}]$ and $B_{n\times n} = [b_{ij}]$ is

$$A \oplus B = \begin{bmatrix}A & 0\\ 0 & B\end{bmatrix}_{(m+n)\times(m+n)}.$$

1.4.5 Equality of matrices : Two matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{p\times q}$ are said to be equal if

i) $m = p$, i.e., number of rows in A is equal to the number of rows in B.
ii) $n = q$, i.e., number of columns in A is equal to number of columns in B.
iii) $a_{ij} = b_{ij}$ for $i = 1,2,\dots,m$ and $j = 1,2,\dots,n$.

i.e., order is same and the corresponding entries are also same of both matrices.

Example 1.4.6 $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$ are not equal matrices because their order is not same.

1.4.7 Subtraction of matrices : Let A and B are two matrices of same order. Then the subtraction of matrices is denoted by A–B and $A-B = A+(-B)$, i.e., the difference between two matrices $A_{m\times n} = [a_{ij}]$ and $B_{m\times n} = [b_{ij}]$ is defined as an $m \times n$ matrix

$$A - B \equiv [a_{ij} - b_{ij}].$$

1.4.8 Power of a matrix : The ith power of the $n \times n$ matrix A, denoted by A^i, is defined as follows:

$$A^i = \begin{cases} \prod_{j=1}^{i} A & \text{for positive integers } i\\ I_n & \text{for } i = 0\\ \left(\prod_{j=1}^{-i} A\right)^{-1} & \text{for negative integers } i, \text{if } \det(A) \neq 0\end{cases}$$

In particular, let A be any square matrix. Then,

1. $A^1 = A$ and
2. $A^{n+1} = A^n . A, n \in \mathbb{N}$

i.e., $A^2 = A.A$, $A^3 = A^2.A = (A.A).A$ and so on.

3. $A^m A^n = A^{m+n}$
4. $(A^m)^n = A^{mn}$ for all $m, n \in \mathbb{N}$.

1.4.9 Scalar multiplication of a matrix : Let $A = [a_{ij}]_{m\times n}$ be a $m \times n$ matrix and c be any number called a scalar. Then, the matrix obtained by multiplying every element of A by c is called the scalar multiplication of A by c and it is denoted by $cA \equiv [ca_{ij}]$ and $Ac \equiv [ca_{ij}]$.

1.4.10 Properties of Scalar Multiplication :

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are two matrices and k, l are scalars. Then,

1. $k(A+B) = kA+kB$
2. $(k + l)A = kA + lA$
3. $(kl)A = k(lA) = l(kA)$
4. $(-k)A = -(kA) = k(-A)$
5. $1.A = A$
6. $(-1)A = -A$.

1.4.11 Multiplication of matrices : Let A be a matrix of order $m \times n$ and B be a matrix of order $p \times q$. The matrix multiplication is possible if number of columns of A is equal to the number of rows of B, i.e. if $n = p$ and their product AB is a matrix of order $m \times q$.

If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$,then their product AB is a matrix $C = [c_{ij}]_{m\times p}$, where c_{ij} = sum of the product of elements of ith row of A with the corresponding elements of jth column of

$$B = [a_{i1}\ a_{i2}\ \dots\ a_{in}]\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

$$= a_{i1}b_{1j} + b_{i2}b_{2j} + \cdots + b_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Note 1.4.12 The matrices A and B are **conformable** if their product AB is defined, that is, if the number of columns of A is equal to the number of rows of B. In the matrix product AB, the matrix A is said to be **postmultiplied** by B and the matrix B is said to be **premultiplied** by A.

Example 1.4.3 Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$. Then, $AB = \begin{bmatrix} 5 & -1 & 2 \\ 5 & 1 & 3 \\ 6 & -4 & 1 \end{bmatrix}$.

1.4.13 Properties of matrix multiplication :

1. Matrix multiplication is not commutative. (in general).
2. Matrix multiplication is associative, i.e. $(AB)C = A(BC)$, whenever both sides are defined.
3. Matrix multiplication is distributive over matrix addition,
   i.e., $A(B+C) = AB+AC$
   and $(A+B)C = AC+BC$.
4. If A is $m \times n$ matrix and O is a null matrix, then
   i) $A_{m\times n}O_{n\times p} = O_{m\times p}$

ii) $O_{p\times m}\, A_{m\times n} = O_{p\times n}$.

5. In case of matrix multiplication, if AB = O, then it is not necessary that BA = O.
e.g., Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then, AB = O but $BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq O$.
6. If A is an $m \times n$ matrix, then $I_m A = A = A I_n$.
7. If A be complex matrix such that sum of elements of each row is k, then sum of elements of each row of A^m is k^m, where m is positive integer.
8. Let A be a $n \times n$ complex matrix such that sum of each row of A is k. Then sum of all elements of A^m is nk^m.
9. Let A and B are any complex matrices. If sum of elements of each row of A is k_1 and sum of elements of each row of B is k_2, then sum of elements of each row of AB is $k_1 k_2$.
10. If A and B are two $n \times n$ complex matrices such that sum of elements of each row of A is k_1 and sum of elements of each row of B is k_2, then sum of all elements of AB is nk_1k_2.
11. The product of two matrices can be null matrix while neither of them is the null matrix.
e.g., $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but neither A = O nor B = O.
12. If A and B are two square matrices of same order such that AB = BA and n is positive integer, then $(A+B)^n$ can be expanded by binomial theorem,
i.e., $(A+B)^n = {}^nC_0 A^n B^0 + {}^nC_1 A^{n-1} B^1 + \ldots + {}^nC_n A^0 B^n$.
13. If A and B are two square matrices of same order such that AB = O = BA and n is any positive integer, then $(A+B)^n = A^n + B^n$.
14. If A is $m \times n$ matrix and B is $n \times p$ matrix, then rows of AB are the linear combination of rows of B and columns of AB are the linear combination of columns of A.
15. If the product AB exists, it is not necessary that BA also exists. **e.g.,** Let A is a 3×4 and B is a 4×5 matrix, then AB is defined but BA is not defined.
16. If BA exists, it is not necessary that AB exists.
17. AB and BA both need not exist.
18. Both AB and BA may exist but their order need not be same.
19. Both AB and BA may exist and their order may be same but matrices need not be equal, i.e., $AB \neq BA$.
20. Both AB and BA may exist and AB = BA.
21. For any matrix A, there does not exist a matrix B such that AB–BA = I.
22. If a matrix is multiplied by a non singular matrix, the rank of given matrix does not alter.
23. If A, B, C, D are $n \times n$ matrices such that ABCD = I. Then,
$$ABCD = DABC = CDAB = BCDA = I.$$
24. Let A and B be $n \times n$ matrices. Then, $AB = A \pm B \Rightarrow AB = BA$.
25. **Square root matrix**: An $n \times n$ matrix B is a square root of the $n \times n$ matrix A if BB = A. It is denoted by $A^{\frac{1}{2}}$.
26. Let A be an $n \times n$ matrix so that every row and every column have one and only one nonzero entry that is either 1 or – 1 and all other entries are 0. Then, $A^k = I$ for some positive integer k.
27. Let A be an n–square matrix of rank r. If A satisfies $A^2 = A$ but is neither 0 nor I, then for every positive integer k, $1 < k \leq n - r$, there exists a matrix B such that
$AB = BA = 0$ and $(A+B)^{k+1} = (A+B)^k \neq (A+B)^{k-1}$.
28. Let A be an $n \times n$ real or complex matrix. If $A^k = I$ for some positive integer k, then $T^{-1}AT$ is diagonal for some complex matrix T.

1.4.14 Direct product : The direct product or **Kronecker product** or **tensor product** of two matrices $A_{m\times n} = [a_{ij}]$ and $B_{p\times q} = [b_{ij}]$ is $A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}_{mp\times nq}$.

1.4.15 Hadamard product : The Hadamard product or **Schur product** or **elementwise product** of the two matrices $A_{m\times n} = [a_{ij}]$ and $B_{m\times n} = [b_{ij}]$ is defined as an $(m \times n)$ matrix

$$A \odot B \equiv [a_{ij}b_{ij}].$$

1.5 Trace, Determinant and Adjoint of a matrix

1.5.1 Trace : The trace of an $n \times n$ matrix $A = [a_{ij}]$ is $\text{tr } A = \text{tr}(A) \equiv a_{11} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$, i.e., the sum of elements of a square matrix A lying on the principal diagonal is called the trace of A and it is denoted by tr(A) .

1.5.2 Properties of trace

1. Let $A, B \in M_n(\mathbb{C})$ and k is any complex number. Then,
   a) $\text{tr}(kA) = k.\text{tr}(A)$
   b) $\text{tr}(A+B) = \text{tr}(A)+\text{tr}(B)$
   c) $\text{tr}(AB) = \text{tr}(BA)$
   d) $\text{tr}(AB)^k = \text{tr}(BA)^k$
   e) $\text{tr}(ABC) = \text{tr}(BCA)$ for every $C \in M_n(\mathbb{C})$
   f) $\text{tr}(AB) \neq \text{tr}(A).\text{tr}(B)$ (In general)
   g) $\text{tr}[(AB - BA)(AB + BA)] = 0$
   h) $\text{tr}(A') = \text{tr}(A)$, where A' is the transpose of A.
   i) $\text{tr}(A^*) \neq \text{tr}(A)$, where A^* is conjugate transpose of A(In general).
2. Let A be real matrix. Then,
   $\text{tr}(AA') = \text{tr}(A'A) = \sum_{i=1}^{n}\sum_{j=1}^{n} {a_{ij}}^2$ = sum of square of each element of A.
3. Let $A = [a_{ij}]_{n\times n}$ be any complex matrix. Then,
   $\text{tr}(A^*A) = \text{tr}(A A^*) = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2$ = sum of square of modulus of each element of A.
4. Let A be $n \times n$ real matrix. Then, $\text{tr}(AA')$ is a non negative real number.
5. Let A be a $n \times n$ complex matrix. Then, $\text{tr}(AA^*)$ is a non negative real number.
6. Let A be a $n \times n$ complex matrix. Then, $\text{tr}(AA^*) = 0$ iff A is a zero matrix.
7. Let A and B be $m \times n$ matrices. Then, $\text{tr}(A'B) = \text{tr}(BA') = \text{tr}(AB') = \text{tr}(B'A)$.
8. Let $A \in M_n(\mathbb{C})$. Then $A^n = 0$ if $\text{tr } A^k = 0, k = 1,2, \ldots, n$.
9. Let $A, B \in M_n(\mathbb{C})$. If $AB = 0$, then for any positive integer k, $\text{tr}(A + B)^k = \text{tr } A^k + \text{tr } B^k$.
10. Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Then, for any positive integer $k \geq 2$, $\text{tr } A^k = \text{tr } A^{k-1} + \text{tr } A^{k-2}$.
11. If A is a real matrix such that $A^3 + A = 0$, then $\text{tr } A = 0$.

1.5.3 Determinant : Let $A = [a_{ij}]$ be a square matrix of order m. Then determinant of A is denoted by $|A|$ or det(A) or det A or Δ_n.

Determinant of a square matrix of order 1

If $A = [a_{11}]$ is a square matrix of order 1, then the determinant of A is a_{11}, i.e., $|A| = a_{11}$.

Determinant of a square matrix of order 2

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a square matrix of order 2. Then, $|A| = a_{11}a_{22} - a_{12}a_{21}$.

Determinant of a square matrix of order n > 2

The determinant of square matrix of order $n > 2$ is obtained by expanding A about a row or a column by multiplying each element a_{ij} in i^{th} row with $(-1)^{i+j}$ times the determinant of submatrix obtained by leaving the row and column passing through the element, i.e., if we expand the matrix A about first row, then determinant is given as

$\det A = \sum_{j=1}^{n}(-1)^{1+j} a_{1j} \det(\overline{A}_{1j}) = \sum_{j=1}^{n} a_{1j} C_{1j}$,

where the scalar $(-1)^{1+j} \det(\overline{A}_{1j}) = C_{1j}$ is called the **cofactor** of the entry a_{1j} in matrix A.

In other form, the determinant of the $m \times m$ matrix $A = [a_{ij}]$ is defined as

$$\det A = \det(A) \equiv \sum(-1)^p a_{1i_1} a_{2i_2} \dots a_{mi_m},$$

where the sum is taken over all products consisting of precisely one element from each row and each column of A multiplied by – 1 or 1, if the permutation $i_1, \dots, i_m$ is odd or even, respectively.

1.5.4 Determinant of an infinite matrix

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix}$ is a doubly infinite array and if $A_1 = [a_{11}]$, $A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

$A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \dots,$

then if $\lim_{n\to\infty} A_n$ exists, it is called the determinant of A.

1.5.5 Properties of determinant

1. The value of determinant remains unchanged if its rows and columns are interchanged.
2. If any two rows of a determinant are interchanged, then the value of determinant changes by minus sign only.
3. If A and B are square matrices of same order, then $|A+B| \neq |A| + |B|$ (In general).
4. If A and B are two square matrices, then $|AB| = |A|.|B|$.
5. If each element of a row(column) of a determinant is zero, then its value is zero.
6. If each element of a row(column) of a determinant is multiplied by the same constant and then added to the corresponding elements of some other row (column), then the value of the determinant is same.
7. Let $A = [a_{ij}]$ be a square matrix of order n. Then, $|kA| = k^n|A|$.
8. If any two rows of a determinant are identical, then its value is zero.
9. If each element of a row(column) of a determinant is multiplied by constant k, then the value of the new determinant is k times the value of original determinant .
10. If each element of a row(column) of a determinant is expressed as sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants.

e.g., $\begin{vmatrix} a_1 + a_2' & b_1 + b_2' & c_1 + c_2' \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_2' & b_2' & c_2' \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$

11. Let $A = [a_{ij}]$ be a square matrix of order n. Then, the sum of the product of elements of any row (column) with their co-factors is always equal to $|A|$,

    i.e., $\sum_{i=1}^{n} a_{ij} c_{ij} = |A|$ and $\sum_{j=1}^{n} a_{ij} c_{ij} = |A|$.

12. Let $A = [a_{ij}]$ be a square matrix of order n. Then, the sum of the product of elements of any row (column) with the co-factors of the corresponding elements of some other row(column) is zero,

    i.e., $\sum_{j=1}^{n} a_{ij} c_{kj} = 0$ and $\sum_{i=1}^{n} a_{ij} c_{ik} = 0.$

13. Let $f(x) = (p_1 - x)(p_2 - x) \dots (p_n - x)$ and let

$$\Delta_n = \begin{vmatrix} p_1 & a & a & a & \cdots & a & a \\ b & p_2 & a & a & \cdots & a & a \\ b & b & p_3 & a & \cdots & a & a \\ b & b & b & p_4 & \cdots & a & a \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ b & b & b & b & \cdots & p_{n-1} & a \\ b & b & b & b & \cdots & b & p_n \end{vmatrix}.$$

If $a \neq b$, then $\Delta_n = \frac{bf(a) - af(b)}{b - a}$ and

If $a = b$, then $\Delta_n = a \sum_{i=1}^{n-1} f_i(a) + p_n f_n(a)$, where $f_i(a)$ means $f(a)$ with factor $(p_i - a)$ missing.

14. If $a \neq b$, then

$$\begin{vmatrix} a+b & ab & 0 & \cdots & 0 & 0 \\ 1 & a+b & ab & \cdots & 0 & 0 \\ 0 & 1 & a+b & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdots & a+b & ab \\ 0 & 0 & 0 & \cdots & 1 & a+b \end{vmatrix}_{n \times n} = \frac{a^{n+1} - b^{n+1}}{a - b}.$$

15. If $A = \begin{bmatrix} \alpha & 0 & 0 & \cdots & 0 & \beta \\ 0 & \alpha & 0 & \cdots & \beta & 0 \\ \vdots & & \ddots & ⋰ & & \vdots \\ 0 & \beta & 0 & \cdots & \alpha & 0 \\ \beta & 0 & 0 & \cdots & 0 & \alpha \end{bmatrix}_{n \times n}$, where n is even, then $|A| = (\alpha^2 - \beta^2)^{\frac{n}{2}}$.

    Or in other words, if $A = [a_{ij}]_{n \times n}$ and n is even such that $a_{ij} = \begin{cases} \alpha & i = j \\ \beta & i + j = n + 1 \\ 0 & \text{otherwise} \end{cases}$, then $|A| = (\alpha^2 - \beta^2)^{\frac{n}{2}}$.

16. If n is odd and $A = \begin{bmatrix} \alpha & 0 & 0 & \cdots & 0 & 0 & \beta \\ 0 & \alpha & 0 & \cdots & 0 & \beta & 0 \\ \vdots & \vdots & & & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \alpha\ 0 & \beta & \ldots & 0 \\ 0 & 0 & \cdots & 0\ \gamma & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \beta\ 0 & \alpha & \cdots & 0 \\ \vdots & \vdots & & & \vdots & & \vdots \\ 0 & \beta & 0 & \ldots & 0 & \alpha & 0 \\ \beta & 0 & 0 & \ldots & 0 & 0 & \alpha \end{bmatrix}_{n\times n}$, then $|A| = \gamma(\alpha^2 - \beta^2)^{(n-1)/2}$.
17. If A is an $n \times n$ matrix all of whose entries are either 1 or – 1, then $|A|$ is divisible by 2^{n-1}.
18. Let A be an $n \times n$ real matrix. If $A^t = -A$ and n is odd, then $|A| = 0$.
19. Let A, B, C, D be $m \times p$, $m \times q$, $n \times p$, $n \times q$ matrices, respectively, where $m + n = p + q$, where $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is Block matrix. Then,
$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = (-1)^{(mn+pq)} \begin{vmatrix} D & C \\ B & A \end{vmatrix}.$$
In particular, when A, B, C, D are square matrices of the same size,
$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} D & C \\ B & A \end{vmatrix},$$
and for a square matrix A, a column vector x and a row vector y,
$$\begin{vmatrix} A & x \\ y & 1 \end{vmatrix} = \begin{vmatrix} 1 & y \\ x & A \end{vmatrix}.$$
20. Let $A, B, C, D \in M_n(\mathbb{C})$. If matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ has rank n, then $\begin{vmatrix} |A| & |B| \\ |C| & |D| \end{vmatrix} = 0$. Moreover, if A is invertible, then $D = CA^{-1}B$.
21. Let $A, B, C, D \in M_n(\mathbb{C})$ and let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then,
   i) $|M| = |AD^t - BC^t|$ if $CD^t = DC^t$.
   ii) $|M| = |AD^t + BC^t|$ if $CD^t + DC^t = 0$ and if D^{-1} exists.
   where C^t is the transpose of matrix C.
22. Let $A, B, C, D \in M_n(\mathbb{C})$.
   i) If A^{-1} exists, then $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A||D - CA^{-1}B|$.
   ii) If $AC = CA$, then $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|$.
23. Let A and B be $n \times n$ real matrices. Then, $\begin{vmatrix} A & B \\ -B & A \end{vmatrix} \geq 0$.
24. Let A and B be $n \times n$ complex matrices. Then, $\begin{vmatrix} A & B \\ B & A \end{vmatrix} = |A + B||A - B|$.
25. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an invertible matrix with $M^{-1} = \begin{pmatrix} X & U \\ Y & V \end{pmatrix}$, where A and D are square matrices(possibly of different sizes), B and C are matrices of appropriate sizes and X has the same size as A. Then,
   i) $|A| = |V||M|$.
   ii) if A is invertible, then $X = (D - CA^{-1}B)^{-1}$.
   iii) consider a unitary matrix W partitioned as $W = \begin{pmatrix} u & x \\ y & U_1 \end{pmatrix}$, where u is a number and U_1 is a square matrix. Then u and $\det U_1$ have the same modulus, i.e., $|u| = |\det U_1|$.

26. Let $A \in M_n(\mathbb{C})$. Then, there exists an $n \times n$ nonzero matrix B such that $AB = 0$ if and only if $|A| = 0$.
27. Let A be an $(n-1) \times n$ matrix of integers such that the row sums are all equal to zero. Then, $|AA'| = nk^2$ for some integer k, where A' is the transpose of matrix A.
28. Let A be a square matrix such that $|A| = 0$. Then, there exists a positive number δ such that $|A + \varepsilon I| \neq 0$, for any $\varepsilon \in (0, \delta)$.
29. Let x and y be column vectors of n complex components. Then,
    i) $|I - xy^*| = 1 - y^*x$
    ii) $\begin{vmatrix} I & x \\ y^* & 1 \end{vmatrix} = \begin{vmatrix} 1 & y^* \\ x & I \end{vmatrix}$
    iii) if $\delta = 1 - y^*x \neq 0$, then $(I - xy^*)^{-1} = I + \delta^{-1}xy^*$
    iv) $\begin{pmatrix} I & x \\ y^* & 1 \end{pmatrix}^{-1} = \begin{pmatrix} I + \delta^{-1}xy^* & -\delta^{-1}x \\ -\delta^{-1}y^* & \delta^{-1} \end{pmatrix}$,
    where y^* is the complex conjugate of column vector y.

Ques. In property 22, Can B and C on the right-hand side in part ii) be switched ? And does part ii) remain true if the condition $AC = CA$ is dropped ?

Ques. In property 19, is it true in general that $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} A & C \\ B & D \end{vmatrix}$?

Ques. In property 21, show by an example that part ii) is invalid if D is singular and $|M|^2 = |AD^t + BC^t|^2$ for the example constructed.

Example 1.5.6 Find the determinant of the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. (CSIR NET)

Solution : $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

$\therefore\ |A| = (1^2 - 2^2)^{\frac{6}{2}} = (-3)^3 = -27$

$\because$ If $A = [a_{ij}]_{n\times n}$, n is even such that $\begin{cases} a_{ij} = \alpha, & \text{for } i = j \\ a_{ij} = \beta, & \text{for } i + j = n + 1 \\ a_{ij} = 0, & \text{elsewhere} \end{cases} \Rightarrow |A| = (\alpha^2 - \beta^2)^{\frac{n}{2}}$.

1.5.7 Differentiation and Integration of determinants

i) If $\Delta(x) = \begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a_{21}(x) & a_{22}(x) & a_{23}(x) \\ a_{31}(x) & a_{32}(x) & a_{33}(x) \end{vmatrix}$ where $a_{ij}(x)$ are differentiable functions of x, then $\frac{d}{dx}\Delta(x) =$

$$\begin{vmatrix} a'_{11}(x) & a'_{12}(x) & a'_{13}(x) \\ a_{21}(x) & a_{22}(x) & a_{23}(x) \\ a_{31}(x) & a_{32}(x) & a_{33}(x) \end{vmatrix} + \begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a'_{21}(x) & a'_{22}(x) & a'_{23}(x) \\ a_{31}(x) & a_{32}(x) & a_{33}(x) \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a_{21}(x) & a_{22}(x) & a_{23}(x) \\ a_{31}{}'(x) & a_{32}{}'(x) & a_{33}{}'(x) \end{vmatrix}$$

Thus, to differentiate a determinant, we differentiate one row(or column) at a time, keeping others unchanged, i.e., for $n \times n$ matrix, if each $a_{ij}(t)$ be a differentiable function of t, then

$$\frac{d}{dt}\begin{vmatrix} a_{11}(t) & \dots & a_{1j}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & \dots & a_{2j}(t) & \dots & a_{2n}(t) \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1}(t) & \dots & a_{nj}(t) & \dots & a_{nn}(t) \end{vmatrix} = \sum_{j=1}^{n} \begin{vmatrix} a_{11}(t) & \dots & \frac{d}{dt}a_{1j}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & \dots & \frac{d}{dt}a_{2j}(t) & \dots & a_{2n}(t) \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1}(t) & \dots & \frac{d}{dt}a_{nj}(t) & \dots & a_{nn}(t) \end{vmatrix}.$$

ii) If $\Delta(x) = \begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ where $a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$ are constants, then

$$\frac{d}{dx}\Delta(x) = \begin{vmatrix} a_{11}{}'(x) & a_{12}{}'(x) & a_{13}{}'(x) \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Also, $\int \Delta(x)dx = \begin{vmatrix} \int a_{11}(x)dx & \int a_{12}(x)dx & \int a_{13}(x)dx \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$

Note 1.5.8 If the elements of more than one column or rows are functions of x, then integration can be done only after evaluation of the determinant.

1.5.9 Minor : Let $A = [a_{ij}]$ be a square matrix of order n. Then, the minor of an element a_{ij} is the determinant of the square submatrix of order $(n-1)$ obtained by deleting the ith row and jth column from A,

$$\text{minor}(a_{ij}) = \det \begin{bmatrix} a_{1,1} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{n,n} \end{bmatrix}.$$

1.5.10 Principal minor : The determinant $\det \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix}$ of a principal submatrix of the $n \times n$ matrix $A = [a_{ij}]$ is a principal minor of A for $k = 1, \dots, n$.

Or a minor of A of order k is principal if it is obtained by deleting $n-k$ rows and $n-k$ columns.

1.5.11 Leading principal minor : The leading principal minor of A of order k is the minor of A of order k if it is obtained by deleting the last $n-k$ rows and columns.

Result 1.5.12 Let $A=[a_{ij}]_{n\times n}$ be a matrix. Then,

i) number of all minors of order $r\times r$ is ……….. *

ii) number of all principal minors of order $r\times r$ is ${}^{n}C_{r}$.

iii) number of leading principal minors of $r\times r$ is 1.

Example 1.5.13 Write down all the minors, principal minors and leading principal minors of order 1, 2, 3 of a 3×3 matrix.

Solution: Let $A=\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$.

i) Minors of order 1 are $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$.
Principal minors of order 1 are a_{11}, a_{22}, a_{33}.
Leading principal minor of order 1 is a_{11}.

ii) Minors of order 2 are $\begin{vmatrix} a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}, \begin{vmatrix} a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix}, \begin{vmatrix} a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix}, \begin{vmatrix} a_{12} & a_{13}\\ a_{22} & a_{23}\end{vmatrix}, \begin{vmatrix} a_{11} & a_{13}\\ a_{31} & a_{33}\end{vmatrix}, \begin{vmatrix} a_{11} & a_{12}\\ a_{31} & a_{32}\end{vmatrix},$
$\begin{vmatrix} a_{11} & a_{13}\\ a_{21} & a_{23}\end{vmatrix}, \begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}, \begin{vmatrix} a_{12} & a_{13}\\ a_{32} & a_{33}\end{vmatrix}$.
Principal minors of order 2 are $\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}, \begin{vmatrix} a_{11} & a_{13}\\ a_{31} & a_{33}\end{vmatrix}, \begin{vmatrix} a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}$.
Leading principal minor of order 2 is $\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}$.

iii) Minor of order 3 is $\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$.
There is only one minor of order 3 which acts as principal minor and leading principal minor as well.

1.5.14 Co-factor : Let $A=[a_{ij}]$ be a square matrix of order n. Then, the cofactor C_{ij} of a_{ij} in A is equal to $(-1)^{i+j}$ times the determinant of the submatrix of order $(n-1)$ obtained by leaving ith row and j^{th} column of A. It follows from the definition that
$C_{ij}=$ cofactor of a_{ij} in $A=\mathrm{cof}(a_{ij})=(-1)^{i+j}M_{ij}$, where M_{ij} is minor of a_{ij} in A.

$$C_{ij}=\begin{cases} M_{ij} & \text{if } i+j \text{ is even}\\ -M_{ij} & \text{if } i+j \text{ is odd}\end{cases}$$

1.5.15 Adjoint of a matrix : For $n\geq 2$, the $n\times n$ matrix $\mathrm{adj}A=[\mathrm{cof}(a_{ij})]'$ is the adjoint of the $n\times n$ matrix $A=[a_{ij}]$. Thus, the adjoint of a matrix is the transpose of the matrix formed by the co-factors of A. Here, $\mathrm{cof}(a_{ij})$ is the cofactor of a_{ij}.
For example, for $n=3$,

$$\text{adj}A = \begin{bmatrix} \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} & -\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} & \det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ -\det\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} & \det\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} & -\det\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix} \\ \det\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} & -\det\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix} & \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{bmatrix}'.$$

For $n = 1$, adjA is defined to be 1.

Adjugate of a matrix : The adjoint of a matrix is sometimes called its adjugate.

1.5.16 Properties of adjoint

1. If A be any square matrix of order n, then $A(\text{adj}A) = (\text{adj}A)A = |A|\, I_n$, where I_n is the unit matrix of order n.
2. If A is a non singular matrix of order n, then $|\text{adj}(A)| = |A|^{n-1}$.
3. If A is a non singular matrix of order n, then $|\text{adj}(\text{adj}A)| = |A|^{(n-1)^2}$

   Generalization : If A is a non singular matrix of order n, then $\underbrace{|\text{adj adj} \ldots \text{adj}(A)|}_{\text{k times}} = |A|^{(n-1)^k}$.
4. If A is a non singular matrix of order n, then $\text{adj}\big(\text{adj}(A)\big) = |A|^{n-2}A$.
5. If A and B are two square matrices of same order, then $\text{adj}(AB) = \text{adj}(B)\text{adj}(A)$.
6. If A is a square matrix, then $\text{adj}A' = (\text{adj}A)'$, where A' is the transpose of A.
7. Adjoint of a non singular matrix is non singular.
8. If A is a singular matrix of order n, then $|\,(\text{adj}A)\,| = 0$.
9. $\text{adj}(X A X^{-1}) = X\big(\text{adj}(A)\big)X^{-1}$ for any invertible $X \in M_n(\mathbb{C})$.
10. $\underbrace{|\text{adj adj} \ldots \text{adj}(A)|}_{\text{k times}} = |A|$ when A is 2×2 matrix.
11. If A is Hermitian, so is adj(A).
12. **Compound matrix**: $B = [b_{ij}]$ is a compound matrix of A if b_{ij} is a minor of a given size of A. For example, for a (3×3) matrix $A = [a_{ij}]$, $B = \begin{bmatrix} \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} & \det\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix} & \det\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} \\ \det\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix} & \det\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} & \det\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} \\ \det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} & \det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} & \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \end{bmatrix}$ is a compound matrix of A.
13. **Oscillatory matrix**: A real $(m \times m)$ matrix A is called oscillatory if all its minors of all possible orders are nonnegative and there exists a positive integer k such that all minors of A^k are positive.
14. **Totally nonnegative matrix**: A real square matrix is called totally nonnegative if all its minors (of all possible orders) are nonnegative.
15. **Totally positive matrix**: A real square matrix is called totally positive if all its minors (of all possible orders) are positive.

1.6 Singular and non singular matrices

A square matrix is said to be singular if $|A| = 0$ and a square matrix is said to be non singular or regular or invertible if $|A| \neq 0$.

Or matrix A is nonsingular if $A = \big(a_{ij}\big) \in M_n(\mathbb{C})$ satisfies $|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|$, $i = 1,2, \ldots, n$.

1.7 Inverse of a matrix

Let A be a square matrix of order n. If there exists a square matrix B of order n such that

$$AB = BA = I_n,$$

then the matrix A is said to be **invertible** and matrix B is called inverse of matrix A and it is denoted by A^{-1}, i.e., we can write also $AA^{-1} = A^{-1}A = I_n$.

Left inverse matrix : Let A be an $n \times m$ matrix. An $m \times n$ matrix B satisfying $BA = I_m$ is called a left inverse of A.

Right inverse matrix : Let A be an $n \times m$ matrix. An $m \times n$ matrix B satisfying $AB = I_n$ is called a right inverse of A.

1.7.1 Properties of inverse

1. Inverse of a square matrix, if exists is unique.
2. A square matrix A is invertible iff A is non singular, i.e., $|A| \neq 0$.
3. If A is non singular, then $A^{-1} = \frac{\text{adj}A}{|A|}$.
4. If A is non singular, then $(\text{adj}A)^{-1} = \text{adj}A^{-1}$.
5. If A is non singular, then $(A')^{-1} = (A^{-1})'$.
6. If A is non singular matrix, then $|A^{-1}| = \frac{1}{|A|}$.
7. A singular matrix can not have inverse.
8. If A is non singular matrix, then $(A^k)^{-1} = (A^{-1})^k$, where k is any positive integer.
9. If A and B be non singular matrices, then $(AB)^{-1} = B^{-1}A^{-1}$.
10. A square matrix A is singular if $AB - BA = A$.
11. If $A_1, A_2, \dots, A_n$ be non singular matrices of same order, then
$$(A_1 . A_2 \dots A_n)^{-1} = A_n^{-1} \dots A_2^{-1} . A_1^{-1}.$$
12. If the matrix B is inverse of A, then A is inverse of B.
13. For the products AB and BA both to be defined and be equal, it is necessary that A and B are both square matrices of the same order. Thus, non square matrices cannot possess inverse.
14. Let A and B be real matrices such that $A + iB$ is non singular. Then, there exists a real number t such that $A + tB$ is non singular.
15. A real $(m \times m)$ matrix $A = [a_{ij}]$ is said to be an **M–matrix** or **Minkowski matrix** if A is non singular, $a_{ij} \leq 0$ for $i \neq j$ and $A^{-1} \geq 0$.
16. **Schur complement of a matrix :** For matrices $A(m \times m)$ non singular, $B(m \times n)$, $C(n \times m)$, $D(n \times n)$, the $(n \times n)$ matrix $S = D - CA^{-1}B$ is called the Schur complement of A in $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$.
17. **Generalized inverse matrix:** An $(n \times m)$ matrix A^{-} is a generalized inverse of the $(m \times n)$ matrix A if it satisfies $AA^{-}A = A$. Sometimes, it is called **g – inverse** matrix or **effective inverse** matrix or **pseudo – inverse** matrix or **conditional inverse** matrix.
18. **Reflexive generalized inverse matrix:** The $(n \times m)$ matrix A_r^{-} is a reflexive generalized inverse of the $(m \times n)$ matrix A if $AA_r^{-}A = A$ and $A_r^{-}AA_r^{-} = A_r^{-}$.
19. **Moore – Penrose (generalized) inverse matrix:** The $(n \times m)$ matrix A^{+} is the Moore – Penrose (generalized) inverse of the $(m \times n)$ matrix A if it satisfies the following four conditions:
(i) $AA^{+}A = A$, (ii) $A^{+}AA^{+} = A^{+}$, (iii) $(AA^{+})^H = AA^{+}$, (iv) $(A^{+}A)^H = A^{+}A$.
And **MP – inverse** is short form for Moore – Penrose inverse matrix.

20. **EP – matrix :** An $(m \times m)$ matrix A is said to be an EP – matrix if it commutes with its Moore – Penrose inverse A^+, that is, $AA^+ = A^+A$.
21. **Group inverse of a matrix :** If the $(m \times m)$ matrix A satisfies $\text{rank}(A) = \text{rank}(A^2)$, the matrix $A^\#$ satisfying $A^2A^\# = A, A^\#AA^\# = A^\#$ and $AA^\# = A^\#A$ is called the group inverse of A.
22. **Drazin inverse :** Let A be an $n \times n$ matrix with index r, that is, r is the smallest number such that $\text{rank}(A^r) = \text{rank}(A^{r+1})$. The Drazin inverse or D–inverse A^D of A is the unique solution of the set of equations
$$A^{r+1}A^D = A^r$$
$$A^DAA^D = A^D$$
$$AA^D = A^DA.$$
23. Matrix A is non singular if $A = (a_{ij}) \in M_n(\mathbb{C})$ satisfies $|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, i = 1,2, \dots, n.$
24. Let A and B be, respectively, $m \times p$ and $m \times q$ matrices such that $A^*B = 0$, where $p + q = m$. If $M = (A, B)$ is invertible, Then, $M^{-1} = \begin{pmatrix} (A^*A)^{-1}A^* \\ (B^*B)^{-1}B^* \end{pmatrix}$.
25. Assuming that all matrix inverses involved below exist, then
$$(A - B)^{-1} = A^{-1} + A^{-1}(B^{-1} - A^{-1})^{-1}A^{-1}.$$
In particular, $(I + A)^{-1} = I - (A^{-1} + I)^{-1}$ and $|(I + A)^{-1} + (I + A^{-1})^{-1}| = 1$.
26. Assuming that all matrix inverses involved below exist, then
$$(A + iB)^{-1} = B^{-1}A(A + AB^{-1}A)^{-1} - i(B + AB^{-1}A)^{-1}.$$
27. Let $A, B, C, D \in M_n(\mathbb{C})$. If AB and CD are Hermitian. Then,
$$AD - B^*C^* = I \Rightarrow DA - BC = I.$$
28. Let $C = A + B$ and $D = A - B$. If C and D are invertible, then
$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} C^{-1} + D^{-1} & C^{-1} - D^{-1} \\ C^{-1} - D^{-1} & C^{-1} + D^{-1} \end{pmatrix}.$$
29. Let S be the backward identity matrix, that is,
$$S = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}_{n\times n}.$$
Then, $S^{-1} = S^t = S$, where S^t is the transpose of matrix S.
30. Let the given correspondences between complex numbers and real matrices and between complex number pairs and complex matrices:
$$z = x + iy \sim Z = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \in M_2(\mathbb{R}),$$
$$q = (u, v) \simeq Q = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \in M_2(\mathbb{C}).$$
Then,
a) $\bar{z} \sim Z^t$, where $\bar{z}$ denotes the conjugate of z.
b) $ZW = WZ$, where $w \sim W$.
c) $z \sim Z$ and $w \sim W$ imply $zw \sim ZW$.
d) $Z^{-1} = \frac{1}{x^2+y^2}\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$.
e) $Z = P\begin{pmatrix} x + iy & 0 \\ 0 & x - iy \end{pmatrix}P^*$, where $P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$.
f) $|Q| \geq 0$.

g) replace each entry z of Q with the corresponding 2×2 real matrix Z to get $R = \begin{pmatrix} U & V \\ -V^t & U \end{pmatrix} \in M_4(\mathbb{R})$. Then, $|R| \geq 0$.

h) R in part g) is similar to a matrix of the form $\begin{pmatrix} U & X \\ -X & U \end{pmatrix}$.

i) R in part g) is singular if and only if Q is singular and if and only if $u = v = 0$.

31. Let m and n be positive integers and denote $K = \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}$. Let S_K be the collection of all $(m + n)$-square complex matrices X such that $X^*KX = K$.

a) If $A \in S_k$, then A^{-1} exists and $A^{-1}, A^t, A, A^* \in S_K$.

b) If $A, B \in S_k$, then $AB \in S_K$.

Ques. In property 31, what about kA or A + B? Also, discuss a similar problem with $K = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$.

Ques. In property 30, find Z^n for $z = r(\cos\theta + i\sin\theta)$, $r, \theta \in \mathbb{R}$. What is the matrix corresponding to $z = i$? And find Q^{-1} when $|u|^2 + |v|^2 = 1$.